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Weak Circulation Theorems as a Way of Distinguishing between Generalized Gravitation Theories

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We proved in a previous paper that a generalized circulation theorem characterizes Einstein's theory of gravitation as a special case of a more general theory of gravitation, which is also based on the principle of equivalence. Here we pose the question of whether it is possible to weaken this circulation theorem in such ways that it would imply more general theories than Einstein's. This problem is solved. Principally, there are two possibilities. One of them is essentially Weyl's theory.

1. INTRODUCTION

We shall deal here with a problem related to the motion of a continuum consisting of freely gravitating, noncolliding, identifiable particles. The differential constants of motion (DCMs) along the histories of these particles, each of which is carrying a clock, were defined and found in Enosh and Kovetz (1978) in the frame of Einstein's theory of gravitation. The values of these quantities depend on the parametrization of the (identifiable) particles. From among the DCMs of the first order (a concept defined there) in Einstein's theory, we concentrate here on the

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K_{AB} [defined by equation (2.27) of Enosh and Kovetz, 1978], which are of outstanding importance. For example, their derivatives form a basis for those DCMs which are independent of the zero adjustment of the time along the particles' world lines, and they have completely analogous quantities, the $K_{AB}^{(N)}$ [defined by equation (3.1) of Enosh and Kovetz, 1977], among the DCMs of Newton's theory of gravitation. These properties are discussed in Enosh and Kovetz (1978). The importance of the K_{AB} is further emphasized by a fact proved in Enosh and Kovetz (1972), where they appear as $K_{AB}=(r^A\cdot v^B)-(r^B\cdot v^A)$. It was proved there that *the fact that the K_{AB} are DCMs characterizes Einstein's theory of gravitation* as a particular case of a generalized theory of gravitation, based on the principle of equivalence and on some reasonable physical evidence. In Section 2 we shall briefly outline this theory and introduce the physical system involved, including definition of the K_{AB}, in more detail. The meaning of the above statement is as follows. *Given a certain space-time in the frame of the generalized theory, this space-time is included in Einstein's theory if and only if for every parametrization of the particles, all the K_{AB} are constant along every particle's world line in this space-time*. (In fact, we could claim either "all the K_{AB}" or "for every parametrization"; these are equivalent. Our formulation, however, is the most suitable for what follows.)

As was claimed in Enosh and Kovetz (1972), the conservation of the K_{AB} along the particles' motion has an equivalent integral form. This is the vorticity theorem (conservation of circulation, known particularly from hydrodynamics) in the Newtonian frame, and the conservation of the kinematical circulation (Synge, 1937) in the frame of Einstein's theory. Therefore we may refer to the conservation of the K_{AB} as a generalized circulation theorem; this theorem, as we said, characterizes Einstein's theory of gravitation.

We may ask now whether it is possible to weaken this circulation theorem such that its imposition would not imply Einstein's theory, but, rather, permit more general theories. More specifically, instead of claiming that *all* the K_{AB} are DCMs we ask whether nontrivial functions of the K_{AB} exist such that the requirement that they be DCMs (that is, constant along every free particle's motion, for every parametrization of the particles) does not imply Einstein's theory. *A priori* it could happen that every nontrivial DCM of the type $F(K_{AB})$ characterizes Einstein's theory. This, however, is not the case, and, therefore, it is natural (and not trivial) to discuss the problem: which gravitation theories can be characterized by DCMs of the type $F(K_{AB})$? This we do in Section 3. The essential difference in the mathematical technique between the work done in Enosh and Kovetz (1977, 1978) and the present work is as follows. In Enosh and Kovetz (1972, 1978) we looked for DCMs common to a given class of space-times. Therefore, while checking a certain differential quantity we vary the particles' parametrization as well as the space-times in the class. Here, mostly, we are given a differential quantity and have to find the space-times in which it is a DCM. (Then we choose certain differential quantities.) Therefore, while checking a certain space-time we only have to vary the particles' parametrization.

Throughout this paper small Latin and Greek and capital Latin indices run over the ranges $\{0,1,2,3\}$, $\{1,2,3\}$, and $\{\bar{1},\bar{2},\ldots,\bar{6}\}$, respectively (except when otherwise noted). The metric tensor g_{ij} has the signature of $\eta_{ij} \equiv \mathrm{diag}(+1,-1,-1,-1)$, $\{{}^{i}_{jk}\}$ are the Christoffel symbols, $\Gamma^{i}_{jk} = \Gamma^{i}_{kj}$ denote the coefficients of a symmetric affine connection. Γ-covariant derivatives with respect to a parameter, are denoted by means of δ/δ (e.g., $\delta U^{i}/\delta s$), and with respect to coordinates by means of a double stroke (e.g., $g_{ij||k}$). Round and square brackets around indices denote the symmetric and the antisymmetric part, respectively. The curvature tensor, $R^{i}{}_{jkl}$, constructed out of the Γ^{i}_{jk} is chosen so that $2\ \xi^{i}_{||[j||k]} = R^{i}{}_{ajk}\xi^{a}$. The general summation convention is strictly kept: a letter occurring twice, no matter where, as an index in a product should be automatically summed over the whole range of the index. For scalar products between 4-vectors we sometimes write $(AB) \equiv A_{i}B^{i}$ or $(CB) \equiv g_{ij}C^{i}B^{j}$. As usual we mark the important equations by a running number. In addition, however, we introduce in some sections a notation by letters for equations of "local" importance.

We shall have to solve systems of homogeneous linear partial differential equations of the first order for a single unknown function. We shall apply to them the usual technique of crossing processes outlined, for example, in Schouten (1954). In order to describe our operations economically we introduce the following notation: Let $F(y)$ satisfy

$$\text{(a)}: \quad a^{i}\frac{\partial}{\partial yi}F = 0$$

$$\text{(b)}: \quad b^{i}\frac{\partial}{\partial yi}F = 0$$

(Here: $i, j = 1, \ldots, N$; N arbitrary). Then F also satisfies

$$\text{(c)}: \quad a^{j}\frac{\partial}{\partial yj}\left(b^{i}\frac{\partial}{\partial yi}F\right) - b^{j}\frac{\partial}{\partial yj}\left(a^{i}\frac{\partial}{\partial yi}F\right) \equiv C^{i}\frac{\partial}{\partial yi}F = 0$$

obtained, we say, by crossing of (a) and (b). We shall write symbolically $[\mathrm{a},\mathrm{b}] = (\mathrm{c})$.

Before continuing we would like to make the following remark. Although, for obvious physical reasons, the discussion here is carried out for a normal hyperbolic four-dimensional space-time, the results obtained hold in a general Riemannian space (signature arbitrary); one can get them by slight modification of the proofs.

2. THE PHYSICAL SYSTEM AND ITS MATHEMATICAL STRUCTURE

The following result was proved in Enosh and Kovetz (1972). Let g_{ij} be any metric tensor of a normal hyperbolic type defined on a differential manifold. Then there exists a one-to-one correspondence between the projective structures (geometries of paths which are locally geodesics with respect to affine connections), the paths of which can be classified as timelike, null, and spacelike with respect to the metric, and the tensor fields e_{ijk} which satisfy

$$e_{(ijk)}=e_{[ij]k}=0 \tag{2.1}$$

These projective structures are determined by the affine connection

$$\Gamma^i_{jk}=\left\{ {i \atop jk} \right\}+2g^{ia}e_{jka} \tag{2.2}$$

which is specified in the projective class of affine connections by the property that the metrical length s is the affine parameter along the Γ geodesics. An equivalent form of (2.2) [and (2.1)] is

$$g_{ij\|k}=2e_{ijk} \tag{2.3}$$

As a physical system we assume that the timelike and null members of the projective structure form the histories of free gravitating particles and light rays, respectively, and the metric determines the clocks' rate along timelike lines. We adopt here this model for space-time. Einstein's theory follows as a particular case: $e_{ijk}=0$.

Let x^i be arbitrary coordinates in space-time. Six parameters $(d)\equiv(d^A)$ serve to identify the possible motions of free particles: $x^i=\bar{x}^i(s,d)$ [$x=\bar{x}(s;d)$ below], where s is the proper time. We define 13 vectors at $\bar{x}(s;d)$:

$$U^i\equiv\frac{\partial\bar{x}^i}{\partial s}, \qquad D^i_A\equiv\frac{\partial\bar{x}^i}{\partial d^A}, \qquad U^i_A\equiv\frac{\delta}{\delta d^A}U^i\equiv\frac{\delta}{\delta s}D^i_A \tag{2.4}$$

They satisfy

$$(UU)=1, \qquad (UU_A)=-e_{ijk}U^iU^jD^k_A \tag{2.5}$$

$$\frac{\delta}{\delta s}U^i=0 \tag{2.6}$$

The second equation of (2.5) is a consequence of the first one $[(UU)=1]$, and (2.3). Let us denote

$$(U^i)=(U^0,U^1,U^2,U^3), \qquad (D_A^i)=(D_A^0, D_A^1, D_A^2, D_A^3),$$

$$(U_A^i)=(U_A^0,U_A^1,U_A^2,U_A^3).$$

Then a necessary and sufficient condition for $\bar{x}(s;d)$ to include (locally) all the free particles' motions is

$$\det\begin{bmatrix} (U^i) & 0 \\ (D_{\bar{1}}^i) & (U_{\bar{1}}^i) \\ \vdots & \vdots \\ (D_{\bar{6}}^i) & (U_{\bar{6}}^i) \\ 0 & (U^i) \end{bmatrix} \neq 0 \tag{2.7}$$

Let us denote the matrix appearing in (2.7) by x and the metrical matrix by (g_{ij}). It is possible to nullify the first column and the first row, apart from their last components, of the matrix

$$x\begin{pmatrix} 0 & -(g_{ij}) \\ (g_{ij}) & 0 \end{pmatrix}x^t$$

(x^t is the transpose of x), by adding suitable products of its last row and its last column to its other rows and columns. It then follows that (2.7) is equivalent to

$$\det(K_{AB})>0 \tag{2.8}$$

where

$$K_{AB}\equiv -P_{ij}D_A^iU_B^j+P_{ij}D_B^iU_A^j\equiv K_{[AB]} \tag{2.9}$$

$$P_{ij}\equiv g_{ij}-g_{ia}g_{jb}U^aU^b \tag{2.10}$$

It is easy to show that the K_{AB} defined by (2.9) are identical with those mentioned in the Introduction.

Now, by means of a construction it is possible to prove the following: *Given a space-time structure (that is, g_{ij}, e_{ijk}), certain $(s_0;d_0)$, $\bar{x}(s_0;d_0)$, and a system of coordinates near $\bar{x}(s_0;d_0)$, then, apart from (2.5) and (2.7), the* *quantities* $\{U^i, D^i_A, U^i_A\}$ *at* $(s_0; d_0)$ *are arbitrary*. Since (2.7) is inequality, *apart from* (2.5), $\{U^i, D^i_A, U^i_A\}$ *are functionally independent*. The range of values of these quantities is attained by transforming the parametrization (d) and the zero adjustment of clocks: $s \rightarrow s' = s + f(d)$ [$f(d)$ arbitrary].

3. THE CENTRAL THEOREM

We sum up the results of the present paper in one theorem, and the remainder of this section is devoted to its proof. It is important to note that all the functions and equations appearing below are assumed to be defined and to exist in certain (not necessarily the largest possible) domains of independent arguments, quantities, events, orthonormal tetrads, parametrizations of particles, zero time adjustments, etc. This is always assumed without explicit reference; all this in the frame of the generalized theory of space-time based on (2.1) and (2.2).

Theorem . (a) If a nontrivial function of the K_{AB}, $F(K_{AB})$, is a DCM, *then* space-time assumes (at least) one of the following possibilities:

(1) $e_{ijk} = g_{ij}e_k - \frac{1}{2}g_{ik}e_j - \frac{1}{2}g_{jk}e_i$ (Weyl type) everywhere (3.1)

(2) $e_i \equiv \frac{1}{3}g^{ab}e_{abi} = 0$ everywhere (3.2)

(b) If space-time is non-Einsteinian ($e_{ijk} \not\equiv 0$) and $F(K_{AB})$ is a DCM, *then* either (1) F is homogeneous of zero order in the K_{AB}, or (2) $F = \Phi(\det(K_{AB}))$, Φ arbitrary function of one variable. If F is not trivial these possibilities are disjoint; in the first case space-time is always of the Weyl type [satisfying (3.1)], in the second space-time is always of $e_i = 0$ type.

(c) Among the $F(K_{AB})$, $\det(K_{AB})$ is (modulo functional dependence) the only characterization as a DCM of the theory proposing $e_i = 0$.

(d) Among the $F(K_{AB})$, no DCM characterization of the Weyl type theory exists. In particular, every nontrivial $F(K_{AB})$ homogeneous of zero order characterizes as a DCM the Weyl type theory with further restrictions on g_{ij} and e_i.

(e) The claim that all homogeneous functions of zero order, $F(K_{AB})$, are DCMs characterizes the theory in which space-time is of the Weyl type [equation (3.1)] and e_i is closed:

$$e_{[i||j]} = 0 \tag{3.3}$$

Proof. Equations (2.1)–(2.6) imply

$$\frac{\delta}{\delta s} U_A^i = R^i_{\ abc} U^a D_A^b U^c$$

Then, starting with (2.9) it is possible to show

$$\begin{aligned}\dot{K}_{AB} &= \left(P_{lb} R^l_{\ man} - P_{la} R^l_{\ mbn}\right) U^m U^n D_A^a D_B^b \\ &\quad + 2\left(U_A^a D_B^b - U_B^a D_A^b\right) e_{abm} U^m \\ &\quad + e_{mna} U^m U^n \left[U_A^a (UD_B) - U_B^a (UD_A) \right] \qquad (3.4)\end{aligned}$$

where a dot denotes (time) s differentiation along the particles' world lines.

From now on we adopt the convention that all the tensor components appearing are understood with respect to an orthonormal tetrad, the zero member of which is U^i. With this agreement (2.7), (2.9), and (2.10) read

$$\det \begin{bmatrix} (D_1^\alpha) & (U_1^\alpha) \\ \vdots & \vdots \\ (D_6^\alpha) & (U_6^\alpha) \end{bmatrix} \neq 0 \qquad (3.5)$$

$$(D_A^\alpha) \equiv (D_A^1, D_A^2, D_A^3), \qquad (U_A^\alpha) \equiv (U_A^1, U_A^2, U_A^3)$$

$$K_{AB} = D_A^\alpha U_B^\alpha - D_B^\alpha U_A^\alpha, \qquad (3.6)$$

$$P_{ij} = \eta_{ij} - \delta_{i0}\delta_{j0} \qquad (3.7)$$

Also

$$g_{ij} = \eta_{ij}, \qquad U^i = \delta_0^i \qquad (3.8)$$

Equation (3.4) becomes, with the aid of (2.1), (2.5), (3.7), (3.8),

$$\dot{K}_{AB} = \left(P_{l\beta} R^l_{\ m\alpha\eta} - P_{l\alpha} R^l_{\ m\beta n}\right) U^m U^n D_A^\alpha D_B^\beta + 4U_{[A}^\alpha D_{B]}^\beta e_{\alpha\beta 0} \qquad (3.9)$$

All the $\langle D_A^\alpha, U_B^\beta \rangle$ for which the K_{AB}, defined by (3.6), are given constants form a manifold of dimension 21 (=36–15), since the K_{AB} are 15 functionally independent functions of the 36 $\{D_A^\alpha, U_B^\alpha\}$, (Enosh and Kovetz, 1977); we call it a *K manifold.*

A function $F(K_{AB})$ is a DCM in a given space-time if and only if it satisfies $(\partial F/\partial K_{AB})\dot{K}_{AB} = 0$. Specifically with our conventions [with the aid of (3.9) and the remark concluding Section 2]: $F(K_{AB})$ is a DCM in a given space-time (g_{ij} and e_{ijk} given) *if and only if*

$$\frac{\partial F}{\partial K_{AB}}\left(P_{l\beta}R^{l}{}_{m\alpha n}-P_{l\alpha}R^{l}{}_{m\beta n}\right)U^{m}U^{n}D^{\alpha}_{A}D^{\beta}_{B}-4\frac{\partial F}{\partial K_{AB}}D^{\alpha}_{[A}U^{\beta}_{B]}e_{\alpha\beta 0}=0$$

for the $\langle D^{\alpha}_{A},U^{\beta}_{B}\rangle$ in the K manifold determined by the arguments of F. (This equation should be understood in a region of space-time and for all the orthonormal tetrads in a certain domain of tetrads.)

A transformation of the type $D^{\alpha}_{A}\to\lambda D^{\alpha}_{A}, U^{\alpha}_{A}\to\lambda^{-1}U^{\alpha}_{A}$ (λ arbitrary) retains the K manifold. Hence it follows immediately that *$F(K_{AB})$ is a DCM in a given space-time if and only if*

$$\frac{\partial F}{\partial K_{AB}}D^{\alpha}_{[A}U^{\beta}_{B]}e_{\alpha\beta 0}=0 \tag{3.10a}$$

$$\frac{\partial F}{\partial K_{AB}}\left(P_{l\beta}R^{l}{}_{m\alpha n}-P_{l\alpha}R^{l}{}_{m\beta n}\right)U^{m}U^{n}D^{\alpha}_{A}D^{\beta}_{B}=0 \tag{3.10b}$$

for the $\langle D^{\alpha}_{A},U^{\beta}_{B}\rangle$ in the K manifold determined by the arguments of F.

We shall dwell a little more upon equation (3.10a). Since $e_{\alpha\beta 0}=e_{(\alpha\beta)0}$, equation (3.10a) takes the form

$$(\partial F/\partial K_{[AB]})D^{(\alpha}_{A}U^{\beta)}_{B}e_{\alpha\beta 0}=0$$

where $(\partial F/\partial K_{[AB]})$ is understood so that the antisymmetrization is performed after differentiating. (In fact, there can be no other meaning.) A simultaneous orthogonal rotation of the $D^{\alpha}_{A}, U^{\alpha}_{A}$: $D^{\alpha}_{A}\to T^{\alpha\beta}D^{\beta}_{A}, U^{\alpha}_{A}\to T^{\alpha\beta}U^{\beta}_{A}$, ($T^{\alpha\beta}$ an orthogonal matrix) retains the K manifold, and we may apply the lemma of the Appendix to the last equation; then we obtain the following. *Equation* (3.10*a*) *is satisfied if and only if for every sequence of permitted and consistent values of the arguments exactly one of the following four possibilities exists*:

$$(\partial F/\partial K_{[AB]})D^{(\alpha}_{A}U^{\beta)}_{B}=0 \tag{3.11a}$$

$$(\partial F/\partial K_{[AB]})D^{(\alpha}_{A}U^{\beta)}_{B}\neq 0, \qquad e_{\alpha\beta 0}=0 \tag{3.11b}$$

$$(\partial F/\partial K_{[AB]})D^{(\alpha}_{A}U^{\beta)}_{B}\neq 0, \qquad (\partial F/\partial K_{[AB]})D^{\mu}_{A}U^{\mu}_{B}=0,$$

$$e_{\alpha\beta 0}=a\delta_{\alpha\beta} \qquad a\neq 0 \tag{3.11c}$$

$$(\partial F/\partial K_{[AB]})D^{(\alpha}_{A}U^{\beta)}_{B}=a\delta^{\alpha\beta}, \qquad a\neq 0, \qquad e_{\alpha\beta 0}\neq 0, \qquad e_{\mu\mu 0}=0 \tag{3.11d}$$

Proof of part (a). Assume that F is a nontrivial DCM in a given space-time. The K_{AB} are arbitrary antisymmetric; hence $(\partial F/\partial K_{[AB]})\neq 0$. We choose K_{AB} such that $(\partial F/\partial K_{[AB]})\neq 0$. We claim that it is possible to find $\langle D_A^\alpha, U_B^\alpha\rangle$ on the chosen K manifold such that $(\partial F/\partial K_{[AB]})D_A^{(\alpha}U_B^{\beta)}\neq 0$: otherwise, $(\partial F/\partial K_{[AB]})D_A^{(\alpha}U_B^{\beta)}=0$ identically over the K manifold. Then applying to this equation transformations of the type

$$D_A^\alpha \to D_A^\alpha, \qquad U_A^\alpha \to U_A^\alpha + \lambda\delta^{\alpha(\mu}D_A^{\nu)}, \qquad \lambda \text{ arbitrary}.$$

which retain the K manifold, we can obtain $(\partial F/\partial K_{[AB]})D_A^\alpha D_B^\beta=0$. In a similar way $(\partial F/\partial K_{[AB]})U_A^\alpha U_B^\beta=0$. By applying to these equations the transformations $\{D_A^\alpha \to D_A^\alpha+\lambda U_A^\alpha, U_A^\alpha\to U_A^\alpha\}$ which too retain the K manifold, we obtain $(\partial F/\partial K_{[AB]})D_A^{[\alpha}U_B^{\beta]}=0$, and finally with the first equation we have $(\partial F/\partial K_{[AB]})D_A^\alpha D_B^\beta=(\partial F/\partial K_{[AB]})D_A^\alpha U_B^\beta=(\partial F/\partial K_{[AB]})U_A^\alpha D_B^\beta=(\partial F/\partial K_{[AB]})U_A^\alpha U_B^\beta=0$, which imply, with the aid of (3.5), $(\partial F/\partial K_{[AB]})=0$. Contradiction. Hence $\langle D_A^\alpha, U_B^\beta\rangle$ exist such that $(\partial F/\partial K_{[AB]})D_A^{(\alpha}U_B^{\beta)}\neq 0$.

Now we choose a point provided with an orthonormal tetrad in space-time. F has to satisfy (3.11) at this point. But, for the above D_A^α, U_A^β equation (3.11a) is untrue; hence one of (3.11b–d) has to be true. This means that $e_{\alpha\beta 0}=0$ or $e_{\alpha\beta 0}=a\delta_{\alpha\beta}$ or $e_{\mu\mu 0}=0$. Since this is true for every tetrad at this point we easily obtain by continuity considerations that an orthonormal tetrad neighborhood exists such that $e_{\mu\mu 0}=0$ for all its members, or $e_{\alpha\beta 0}=a\delta_{\alpha\beta}$ for all its members. In the first case, $g^{ab}e_{abc}U^c=0$, [since $e_{000}=0$ by (2.1)], for a neighborhood of U^c. Hence $g^{ab}e_{abc}=0$, [equation (3.2)]. In the second case $e_{\alpha\beta 0}-\frac{1}{3}g^{ab}e_{ab0}g_{\alpha\beta}=0$ for all the tetrads in the neighborhood. This implies that

$$a_{ijk}\equiv e_{ijk}-g_{ij}e_k+\tfrac{1}{2}g_{ik}e_j+\tfrac{1}{2}g_{jk}e_i$$

where $e_i\equiv\frac{1}{3}g^{ab}e_{abi}$, satisfies

$$a_{(ijk)}=a_{[ij]k}=0, \qquad g^{ab}a_{abk}=0, \qquad a_{abc}n^a m^b U^c=0$$

for a neighborhood of (timelike) U^a and $(nU)=(mU)=0$. We claim that this implies $a_{ijk}=0$ [equation (3.1)]: At first we may replace every n^a by $n^a-(g_{fg}U^fU^g)^{-1}(g_{de}U^dn^e)U^a$, where now n^a is arbitrary, and a similar replacement of m^a. Then with the aid of $a_{(ijk)}=a_{[ij]k}=0$ we obtain

$$a_{ab(c}g_{de)}-a_{a(de}g_{c)b}-a_{b(de}g_{c)a}=0$$

Now some contractions of this equation and the symmetries of a_{abc} lead to $a_{abc}=0$, as was claimed. Thus at all points (since our chosen point was arbitrary), (3.1) or (3.2) exist. We have to show, however, that (3.1) is everywhere true, or the same for (3.2). Assume, on the contrary, that two points exist such that at one (3.1) is true with $e_i \neq 0$, and at the second (3.2) is true with $e_{ijk} \neq 0$. It is possible to find an orthonormal tetrad at the first point such that $e_{\alpha\beta 0} = a\delta_{\alpha\beta}$, $a \neq 0$. It is possible, also, to find an orthonormal tetrad at the second such that $e_{\alpha\beta 0} \neq 0$ [since, otherwise, $e_{\alpha\beta 0} = 0$ holds always and, by the preceding considerations, implies equation (3.1); but equations (3.1) and (3.2) together imply the vanishing of e_{ijk}]. We know already that D_A^α, U_B^β exist such that $(\partial F/\partial K_{[AB]})D_A^{(\alpha}U_B^{\beta)} \neq 0$. Since (3.11) has to exist at both points we find $(\partial F/\partial K_{[AB]})D_A^\mu U_B^\mu = 0$ owing to the first point and $(\partial F/\partial K_{[AB]}D_A^{(\alpha}U_B^{\beta)} = a\delta^{\alpha\beta}$ owing to the second. Hence $(\partial F/\partial K_{[AB]}D_A^{(\alpha}U_B^{\beta)} = 0$. Contradiction. This completes the proof of part (a). ∎

Proof of part (b). Assume that space-time is (given) non-Einsteinian and $F(K_{AB})$ is a DCM. Assume, further, that F is nontrivial. [Otherwise, the argument is trivial.] It is obvious, then, that the possibilities 1, F homogeneous of zero order, and 2, $F = \Phi(\det(K_{AB}))$, are, indeed, disjoint. We now make use of part (a). Hence, space-time has to be of the Weyl type or of the $e_i = 0$ type.

Assume first that e_{ijk} takes the form (3.1). An event exists at which $e_{ijk} \neq 0$, and an orthonormal tetrad at this event exists such that $e_{\alpha\beta 0} = a\delta_{\alpha\beta}$ for $a \neq 0$. However, one of the possibilities (3.11) has to be satisfied. Obviously, the relevant possibilities are (3.11a) and (3.11c), and in both of them $(\partial F/\partial K_{[AB]})D_A^\mu U_B^\mu = 0$. With the aid of (3.6) this is equivalent to

$$(\partial F/\partial K_{AB})K_{AB} = 0 \tag{3.12}$$

which means that F is homogeneous of zero order in the K_{AB}.

Assume, next, $e_i = 0$. In a similar way we find, again by (3.11), (3.6) and a part of the considerations in the proof of part (a), that $(\partial F/\partial K_{[AB]})D_A^{(\alpha}U_B^{\beta)} = a\delta^{\alpha\beta}$. Equivalently,

$$\left(D_{[A}^{(\alpha}U_{B]}^{\beta)} - \tfrac{1}{6}\delta^{\alpha\beta}K_{AB}\right)(\partial F/\partial K_{AB}) = 0 \tag{3.13}$$

Now we find the solutions of this equation. We look upon $F(K_{AB})$ as a function of the D_A^α, U_B^β according to (3.6). It follows that

$$\left(\partial F/\partial D_A^\beta\right) = (\partial F/\partial K_{AB})U_B^\beta - (\partial F/\partial K_{BA})U_B^\beta$$

Therefore

$$(\partial F/\partial K_{AB})D_{[A}^{(\alpha}U_{B]}^{\beta)} = \tfrac{1}{4}\left[D_A^\alpha(\partial F/\partial D_A^\beta) + D_A^\beta(\partial F/\partial D_A^\alpha)\right] \tag{3.14a}$$

$$(\partial F/\partial K_{AB})K_{AB} = D_A^\alpha(\partial F/\partial D_A^\alpha) \tag{3.14b}$$

Now, $F(K_{AB})$ satisfies (3.13) *if and only if* as a function of the D_A^α, U_B^β:

(a): $D_A^\gamma(\partial F/\partial U_A^\eta)+D_A^\eta(\partial F/\partial U_A^\gamma)=0$

(b): $U_B^\nu(\partial F/\partial D_B^\mu)+U_B^\mu(\partial F/\partial D_B^\nu)=0$

(c): $D_C^\kappa(\partial F/\partial D_C^\lambda)-U_C^\lambda(\partial F/\partial U_C^\kappa)=0$

(d): $D_E^\alpha(\partial F/\partial D_E^\beta)+D_E^\beta(\partial F/\partial D_E^\alpha)-\frac{2}{3}\delta^{\alpha\beta}D_E^\rho(\partial F/\partial D_E^\rho)=0$

Equations (a), (b), (c) above are satisfied by every function of the K_{AB} (a simple check); moreover, they ensure that F depends on the D_A^α, U_B^β through the K_{AB}. [Cf. equations (b), (d), (f) of Section (3.3) of Enosh and Kovetz (1977)]. Equation (d) is, in fact, (3.13) (with the aid of (3.14)). We apply now some linear combinations and crossing operations to equations (a)–(d). A contraction of $\alpha=\eta$ in [a, d] leads to

(e): $D_a^\gamma(\partial F/\partial U_A^\eta)=0$

A contraction of $\mu=\nu$ in [e, b] leads to

(f): $D_C^\kappa(\partial F/\partial D_C^\lambda)-U_C^\kappa(\partial F/\partial U_C^\lambda)=0$

Applying (f) and (c) to (d) leads to

(g): $D_E^\alpha(\partial F/\partial D_E^\beta)-\frac{1}{3}\delta^{\alpha\beta}D_E^\rho(\partial F/\partial D_E^\rho)=0$

A contraction of $\mu=\beta$ in $[b, g]$ leads [with further use of (b)] to

(h): $U_B^\nu(\partial F/\partial D_B^\mu)=0$

The system (e), (h), (f), (g) implies and is implied by the original system (a), (b), (c), (d). It is possible to show that the new system is closed, that is, crossing operations do not lead to any new equations. Equations (e), (h), (f), (g) consist of *at most* 9, 9, 9, 8 linearly independent equations, respectively. They form, however, exactly 35 $(=9+9+9+8)$ linearly independent equations for the 36 unknowns $\{(\partial F/\partial D_A^\alpha), (\partial F/\partial U_B^\beta)\}$ since the addition of only one equation: $D_E^\rho(\partial F/\partial D_E^\rho)=0$ implies, with the aid of (3.5), $(\partial F/\partial D_A^\alpha)=(\partial F/\partial U_B^\beta)=0$. Hence we know that equation (3.13) admits, up to a functional dependence, one and only one solution. We claim that $F_0\equiv\det(K_{AB})$ is the solution of (3.13). By (2.8) $F_0\neq 0$ and obviously

$$(\partial F_0/\partial K_{AB})=(\partial[\det(K_{AB})]/\partial K_{AB})=\det(K_{AB})K^{BA}$$
$$=F_0K^{BA} \tag{3.15}$$

where (K^{AB}) is the matrix reciprocal to (K_{AB}). In order to prove (3.13) for F_0 it is sufficient (since K^{AB} is antisymmetric) to show that

$$K^{AB}D_A^{(\alpha}U_B^{\beta)}=\delta^{\alpha\beta} \tag{3.16}$$

To this end we observe that we may refer to the indices A, B of D_A^α, U_B^β, K_{AB} as covariant indices, while those of K^{AB} are contravariant with respect to transformation of the type $D_A^\alpha \to D_{A'}^\alpha = \Phi_{A'}^A D_A^\alpha$, $U_A^\alpha \to U_{A'}^\alpha = \Phi_{A'}^A U_A^\alpha$. With respect to these transformations equation (3.16) is a scalar equation. We choose that transformation which makes the matrix of (3.5) the unit matrix, that is,

$$D_A^\alpha = \delta_A^{\bar\alpha}, \qquad U_A^\alpha = \delta_A^{\overline{\alpha+3}} \qquad (\alpha = 1,2,3;\, A = \bar 1,\ldots,\bar 6) \tag{3.17}$$

and then, by (3.6),

$$(K_{AB}) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \equiv J, \qquad (K^{AB}) = J^t$$

or, equivalently,

$$K^{AB} = \begin{cases} 1 & B = A + \bar 3,\ A = \bar 1, \bar 2, \bar 3 \\ -1 & A = B + \bar 3,\ B = \bar 1, \bar 2, \bar 3 \\ 0 & \text{otherwise} \end{cases} \tag{3.18}$$

With the aid of (3.17) and (3.18) we may now calculate

$$\begin{aligned} K^{AB} D_A^{(\alpha} U_B^{\beta)} &= \tfrac{1}{2}\left(K^{AB} D_A^\alpha U_B^\beta + K^{AB} D_A^\beta U_B^\alpha \right) \\ &= \tfrac{1}{2}\left(D_{\bar\mu}^\alpha U_{\overline{\mu+3}}^\beta - D_{\overline{\mu+3}}^\alpha U_{\bar\mu}^\beta + D_{\bar\mu}^\beta U_{\overline{\mu+3}}^\alpha - D_{\overline{\mu+3}}^\beta U_{\bar\mu}^\alpha \right) \\ &= \tfrac{1}{2}\left(\delta_\mu^\alpha \delta_\mu^\beta + \delta_\mu^\beta \delta_\mu^\alpha \right) = \delta^{\alpha\beta} \end{aligned}$$

This completes the proof of part (b). ∎

Proof of part (c). We say that a set of DCMs characterizes a theory if every space-time available to the theory admits these DCMs and, on the contrary, every space-time which admits these DCMs belongs to the theory. According to part (b), among the $F(K_{AB})$ only $\det(K_{AB})$ can have the property of characterizing the theory based on $e_i = 0$. The DCM $F_0 \equiv \det(K_{AB})$ indeed implies $e_i = 0$; we have, however, to show that F_0 is a DCM in *every* space-time satisfying $e_i = 0$; equivalently F_0 has to satisfy equations (3.10). The preceding discussion ensures that F_0 satisfies equation (3.10a). We only have to check (3.10b). By a method similar to that used in the proof of part (b), that is, specializing to the case (3.17) and (3.18), we can prove $K^{AB} D_A^\alpha D_B^\beta = 0$. This fact and equation (3.15) imply, indeed, equation (3.10b). This completes the proof of part (c). ∎

Proof of part (d). Assume, on the contrary, that $F(K_{AB})$ is a nontrivial DCM in all space-time satisfying equation (3.1), (Weyl type). By part (b), F is homogeneous of zero order, and it therefore satisfies equation (3.10a) in every space-time of Weyl's type. As a DCM it satisfies (3.10b) too, in every space-time of Weyl's type. We may represent R^i_{jkl} as a sum of Riemann's tensor and a tensor vanishing when e_{ijk} does. Substitution of this into (3.10b) leads to

$$(\partial F/\partial K_{AB})D^{\alpha}_{[A}D^{\beta}_{B]}(e_{00\alpha||\beta}+e_{0\alpha\beta||0})=0$$

It is easy to show (by means of construction) that $(e_{00\alpha||\beta}+e_{0\alpha\beta||0})$ is arbitrary while varying the space-times even if they are subject to equation (3.1). Hence F has to satisfy

$$(\partial F/\partial K_{AB})D^{\alpha}_{[A}D^{\beta}_{B]}=0$$

We treat this equation by a method similar to that applied to equation (3.13). We think of $F(K_{AB})$ as a function of the $D^{\alpha}_{A}, U^{\beta}_{B}$ according to (3.6). We find that F satisfies

(a): $D^{\gamma}_{A}(\partial F/\partial U^{\eta}_{A})+D^{\eta}_{A}(\partial F/\partial U^{\gamma}_{A})=0$

(b): $U^{\nu}_{B}(\partial F/\partial D^{\mu}_{B})+U^{\mu}_{B}(\partial F/\partial D^{\nu}_{B})=0$

(c): $D^{\kappa}_{C}(\partial F/\partial D^{\lambda}_{C})-U^{\lambda}_{C}(\partial F/\partial U^{\kappa}_{C})=0$

(d): $U^{\rho}_{E}(\partial F/\partial U^{\rho}_{E})=0$

(e): $D^{\alpha}_{F}(\partial F/\partial U^{\beta}_{F})=0$

Equations (a), (b), (c) ensure that F is a function of the K_{AB}. Equation (d) expresses the fact that F is homogeneous of zero order, and Equation (e) is equivalent to the last equation implied by (3.10b). Equations (c) and (d) imply

(f): $D^{\kappa}_{C}(\partial F/\partial D^{\kappa}_{C})=0$

A contraction of $\nu=\beta$ in $[e, b]$ and applying (c), (d), (f) to the resulting equation leads to

(g): $D^{K}_{C}(\partial F/\partial D^{\lambda}_{C})=0$

which with the aid of (c) results in

(h): $U^{\rho}_{E}(\partial F/\partial U^{\tau}_{E})=0$

Equations (e) and (h) imply, with the aid of (3.5),

(i): $(\partial F/\partial U^{\tau}_{E})=0$

then a contraction of $\nu=\tau$ in $[i, b]$ leads to

(j): $(\partial F/\partial D_E^\mu)=0$

Therefore F is a trivial function. Contradiction. This completes the proof of part (d). ■

Proof of part (e). By the preceding discussion we obtain the following. In a given space-time all the homogeneous functions of zero order, $F(K_{AB})$, are DCMs *if and only if* space-time is of the Weyl type and Equation (3.10b) is satisfied by all these functions. For given K_{AB}, $(\partial F/\partial K_{[AB]})$, for our functions, is arbitrary apart from the linear restriction (3.12). Hence, by comparing this equation with (3.10b), in which $(\partial F/\partial K_{[AB]})$ has exactly the same restrictions and no more, we obtain

$$D_A^\alpha D_B^\beta\left(P_{l\beta}R^l_{\ m\alpha n}-P_{l\alpha}R^l_{\ m\beta n}\right)U^m U^n=\mu K_{AB}$$

for any μ, as an equivalent form of the claim that (3.10b) is satisfied for all homogeneous $F(K_{AB})$ of zero order. By (3.6)–(3.8) we may write the last equation in the form

$$D_A^a D_B^b\left(P_{lb}R^l_{\ man}-P_{la}R^l_{\ mbn}\right)U^m U^n=\mu\left(D_B^a U_A^b-D_A^a U_B^b\right)P_{ab} \qquad (3.19)$$

which has to be satisfied in every system of coordinates, owing to its covariance. Apart from (2.5), the U^m, D_A^a, U_B^b appearing in (3.19) are functionally independent, according to the final remark of Section 2. The restriction $(UU_A)+e_{ijk}U^i U^j D_A^k=0$ of (2.5) is redundant, however, since for *arbitrary* D_A^a, U_B^b in a domain we may define $\hat{U}_B^a=U_B^a-[(UU_B)+e_{mnl}U^m U^n D_B^l]U^a$, then $D_A^a, \hat{U}_B^b$ satisfy (2.5) and substitution of them in (3.19) leads to the equation obtained from (3.19) by substitution of D_A^a, U_B^b in it. Therefore we may sum up the last conclusions as follows. Define for U^a satisfying $(UU)=1$

$$m_{ab}\equiv\left(P_{lb}R^l_{\ man}-P_{la}R^l_{\ mbn}\right)U^m U^n \qquad (3.20)$$

Then, all the homogeneous functions of zero order $F(K_{AB})$ are DCMs *if and only if* space-time is of the Weyl type and for every m_{ab} defined above and functionally independent D_A^a, U_A^a

$$m_{ab}D^a_{[A}D^b_{B]}=m_{ab}D_A^a D_B^b=\mu P_{ab}D^a_{[A}U^b_{B]}$$

for a certain μ.

Since μ may depend on the D_A^a, U_B^b we cannot conclude directly that the monomials in the variables D_A^a, U_B^b in the last equation vanish separately. We cannot even put $U_A^a=0$ since this is excluded by (2.7). Nevertheless, the left-hand side of the last equation vanishes identically. We prove this. Assume, on the contrary, $m_{ab}D_A^aD_B^b\neq 0$ for the certain D_A^a. Then,

$$m_{ab}D_A^aD_B^b=\mu_0 P_{ab}D_{[A}^aU_{B]}^b, \qquad \mu_0\neq 0$$

and, of course, an index b_0 exists such that $P_{ab_0}D_A^a\neq 0$. Let V_A be a 6-tupple which does not belong to the space spanned by the $\{D_A^a\}_{a=0}^3$, (four dimensional at most). Define: $\hat{U}_A^a\equiv U_A^a+\lambda V_A\delta_{b_0}^a$. Then

$$m_{ab}D_A^aD_B^b=\mu_\lambda P_{ab}D_{[A}^a\hat{U}_{B]}^b=\mu_\lambda P_{ab}D_{[A}^aU_{B]}^b+\lambda\mu_\lambda P_{ab_0}D_{[A}^aV_{B]}$$

or by the preceding equation

$$m_{ab}D_A^aD_B^b=\mu_\lambda\mu_0^{-1}m_{ab}D_A^aD_B^b+\lambda\mu_\lambda P_{b_0a}D_{[A}^aV_{B]}$$

But $P_{b_0a}D_A^a\neq 0$ and the antisymmetric quantity of dimension 6, $P_{b_0a}D_{[A}^aV_{B]}$, is independent of the $\{D_{[A}^aD_{B]}^b\}_{a,b=0}^3$; hence $\lambda\mu_\lambda=0$, which implies $\mu_\lambda=0$, which further implies $m_{ab}D_A^aD_B^b=0$. Constradiction. Therefore, $m_{ab}D_A^aD_B^b=0$ for functionally independent D_A^a, and hence $m_{ab}=0$. Thus, by (3.20) and by (2.10),

$$(g_{at}R^t_{mbn}-g_{bt}R^t_{man})U^mU^n-(g_{ap}g_{qt}R^t_{mbn}-g_{bp}g_{qt}R^t_{man})U^pU^qU^mU^n=0$$

for all U^a (in a domain) satisfying $(UU)=1$. This is an equivalent claim to (3.19). By putting $U^a=(g_{ij}\xi^i\xi^j)^{-1/2}\xi^a$, where ξ^a is arbitrary (in a domain), we finally find that the last equation is equivalent to

$$K_{[ab](pqmn)}=0 \tag{3.21}$$

where

$$K_{abpqmn}\equiv(g_{pq}g_{at}-g_{ap}g_{qt})R^t_{mbn} \tag{3.22}$$

Hence the homogeneous functions $F(K_{AB})$ of zero order are DCMs *if and only if* space-time is of the Weyl type and satisfies (3.21). However, by decomposing R^i_{jkl} as a sum of Riemann's tensor and a term vanishing with e_{ijk}, and substituting this decomposition into (3.21), one finds in the case of Weyl's type space-time [equation (3.1)] that (3.21) is equivalent to (3.3). This completes the proof of the theorem. ■

4. SOME CONCLUDING REMARKS

If one is interested in theories which are special cases of the generalized theory [based on equations (2.1) and (2.2)], such that the permitted tensors e_{ijk} at a point form a linear space which is independent of the point and of the orthonormal tetrad (that is, the components e_{ijk} are restricted in the same way with respect to every orthonormal tetrad, the principle of relativity following the footsteps of the principle of equivalence), then only two nontrivial possibilities are available: the $e_i=0$ type theory [equation (3.2)] and the Weyl-type theory [equation (3.1)]. The reason is that the above requirements mean that the e_{ijk} space has to be a representation space of the proper Lorentz group (L_p), and the above two subspaces are irreducible representation spaces for L_p which span the whole space of the e_{ijk} satisfying (2.1). This result is achieved by investigating the spinorial representation of e_{ijk}. This remark refers to a four-dimensional space of a normal hyperbolic type only.

In Section 3 we proved that nontrivial weak circulation theorems may exist only in space-times which belong to one (at least) of the above theories.

The theory based on equation (3.2) ($e_i=0$), is characterized by a single DCM, $\det(K_{AB})$, and a space-time of this type which is not Einstein's, ($e_{ijk}\not\equiv 0$), never admits other DCMs of the type $F(K_{AB})$. From the physical point of view, measurement of $\det(K_{AB})$ is not practical since it requires the existence of a six-parameter set of particles, while it seems that at most a three-parameter set of particles is available.

The theory based on equation (3.1) proposes space-times of the so-called Weyl type. The reason is that their projective and conformal structure form a Weyl's space. (See, for example, [Enosh and Kovetz, 1973]). They have, however, an extra structure—an exact determination of the metric. In such non-Einsteinian ($e_i\neq 0$) space-times one sometimes can find DCMs of the type $F(K_{AB})$, which are always homogeneous of zero order. On the contrary, every nontrivial DCM of this type implies Weyl-type space-times. But no such a DCM characterizes this theory: every one of them restricts the possibilities more than by (3.1) only. Since we had no physical reason to prefer such a function or functions, we tried to choose them all as DCMs. This claim characterizes the Weyl-type space-times with closed e_i [equation (3.3)]. In such space-times, we know from Weyl's theory, the geodesics (free particles and light rays), are Riemannian, derived from a metric g'_{ij}; the rate of clocks, however, is determined by another metric g_{ij} conformal to the metric g'_{ij}. Thus we have arrived at a Riemannian theory of space-time admitting one scalar function.

The homogeneous functions of order zero of the K_{AB} are restricted by only one equation, (3.12). Hence it follows easily that they together with the function $\det(K_{AB})$ span functionally the space of all the functions of the K_{AB}.

If one is interested only in the projective and the conformal structures of space-time (cf. Ehlers, Pirani, and Schild, 1972), only those functions which are invariant under the conformal transformations of the metric are important. Generally the K_{AB} do depend on such transformations. However, it follows by direct calculation that the homogeneous functions of order zero of the K_{AB} are projective–conformal quantities and, by the remark of the preceding paragraph, that up to functional dependence there are no other projective conformal quantities among the functions of the K_{AB}. [$\det(K_{AB})$ should depend on the conformal transformation.] A complete set for them is $[\det(K_{CD})]^{-1/6}K_{AB}$. In the case of conformal–projective space-time in which the projective lines preserve their conformal character, it follows by the main theorem of this paper that the conservation of one (arbitrary) of the above quantities can occur only in a Weyl space and that the conservation of all of them characterizes (locally) the Riemannian structure (up to a multiplying constant of the metric). An interesting question in this context (and also in the more general metrical case) is whether it is possible to weaken the last claim in ways that theories more general than the Riemannian one are characterized, and in case that the answer is yes, to classify the possibilities. This problem is now under consideration.

APPENDIX: A LEMMA CONCERNING MATRICES

Lemma. Let S and T be two real symmetric $(n\times n)$ matrices which satisfy

$$\operatorname{tr}(SRTR^t)=0 \tag{A.1}$$

for every orthogonal matrix R in a certain (arbitrary) neighborhood of the unit matrix I. (R^t stands for the transpose of R.) Then, one of the following possibilities should occur:

(a) $S=0$ or $T=0$

(b) $S\neq 0$, $T\neq 0$, $\operatorname{tr} T=0$, $S=\alpha I$ $\quad(\alpha\neq 0)$

(c) $S\neq 0$, $T\neq 0$, $\operatorname{tr} S=0$, $T=\alpha I$ $\quad(\alpha\neq 0)$

Proof. If the orthogonal matrix R were arbitrary the proof would be much simpler. In such circumstances it is possible to reduce the lemma to diagonal S, T, since every real symmetric matrix is equivalent by an orthogonal matrix to a diagonal matrix and $\operatorname{tr}(AB)=\operatorname{tr}(BA)$. Then the lemma follows as a consequence of some choices of R which change the order of two elements in the diagonal of T ($\det R=-1$(!)].

However, we consider now only those Rs confined to a certain neighborhood of I. Let A, B be arbitrary antisymmetric real matrices. Then $R(\lambda, \mu) = \exp(\lambda A)\exp(\mu B)$ is orthogonal, for all real λ, μ. We put $R = R(\lambda, \mu)$ in (A.1) for both λ and μ in a certain neighborhood of 0 such that $R(\lambda, \mu)$ is contained in the given neighborhood of I. Now we differentiate (A.1) with respect to λ, obtaining one equation, and differentiate, then, the resulting equation with respect to μ, obtaining a second equation. A substitution of $\lambda = \mu = 0$ in (A.1) and in the last two equations leads to

$$\operatorname{tr}(ST) = 0 \tag{A.2}$$

$$\operatorname{tr}(TSA - STA) = 0 \tag{A.3}$$

$$\operatorname{tr}(ABTS + STBA - SATB - SBTA) = 0 \tag{A.4}$$

for every antisymmetric A and B. Since $S = S^t$, $T = T^t$, $A = -A^t$ we obtain with the aid of equation (A.3)

$$\operatorname{tr}(STA) = \operatorname{tr}(STA)^t = -\operatorname{tr}(ATS) = -\operatorname{tr}(TSA) = -\operatorname{tr}(STA)$$

Therefore

$$\operatorname{tr}(STA) = \operatorname{tr}(TSA) = 0 \tag{A.5}$$

for every antisymmetric A. A complete set for the antisymmetric matrices is formed by the matrices $\{(A_{\alpha\beta})\}_{\alpha,\beta=1,\ldots,n}$ defined by

$$(A_{\alpha\beta})_{\mu\nu} = \delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu} \tag{A.6}$$

We put $A = (A_{\alpha\beta})$ in (A.5);

$$0 = \operatorname{tr}[ST(A_{\alpha\beta})] = S_{\nu\rho}T_{\rho\mu}(A_{\alpha\beta})_{\mu\nu} = S_{\beta\rho}T_{\rho\alpha} - S_{\alpha\rho}T_{\rho\beta} = S_{\beta\rho}T_{\rho\alpha} - T_{\beta\rho}S_{\rho\alpha}$$

Therefore

$$ST = TS \tag{A.7}$$

We put now $A = (A_{\alpha\beta})$, $B = (A_{\gamma\delta})$ in equation (A.4) and, with the aid of (A.7), we obtain

$$T_{\delta\mu}S_{\mu\alpha}\delta_{\gamma\beta} - T_{\gamma\mu}S_{\mu\alpha}\delta_{\delta\beta} - T_{\delta\mu}S_{\mu\beta}\delta_{\gamma\alpha} + T_{\gamma\mu}S_{\mu\beta}\delta_{\delta\alpha} - S_{\delta\alpha}T_{\beta\gamma} + S_{\delta\beta}T_{\alpha\gamma} + S_{\gamma\alpha}T_{\beta\delta} - S_{\gamma\beta}T_{\alpha\delta} = 0 \tag{A.8}$$

By the contraction of $\beta=\delta$ and, with the aid of (A.2) and (A.7), we obtain

$$-nS_{\alpha\mu}T_{\mu\gamma}+S_{\mu\mu}T_{\alpha\gamma}+T_{\mu\mu}S_{\alpha\gamma}=0 \tag{A.9}$$

Contraction of $\alpha=\gamma$ in (A.9) leads, with the aid of (A.2), to

$$S_{\mu\mu}T_{\nu\nu}=0 \tag{A.10}$$

We distinguish three cases which cover according to (A.10) all the possibilities (1) $S_{\mu\mu}=T_{\mu\mu}=0$; (2) $S_{\mu\mu}\neq 0$, $T_{\nu\nu}=0$; (3) $T_{\nu\nu}\neq 0$, $S_{\mu\mu}=0$.

In case (1) the following equations are satisfied:

$$-S_{\alpha\delta}T_{\beta\gamma}+S_{\beta\delta}T_{\alpha\gamma}+S_{\alpha\gamma}T_{\beta\delta}-S_{\beta\gamma}T_{\alpha\delta}=0 \tag{A.11a}$$

$$S_{\alpha\mu}T_{\mu\gamma}=0 \tag{A.11b}$$

$$S_{\mu\mu}=T_{\nu\nu}=0 \tag{A.11c}$$

[Equation (A.11b) is, in fact, (A.9), and (A.11a) is a consequence of a substitution from (A.9) to (A.8)]. The only solutions of the system (A.11) (for symmetric $S_{\alpha\beta}, T_{\alpha\beta}$) have the form $S_{\alpha\beta}=0$, $T_{\nu\nu}=0$ or $T_{\alpha\beta}=0$, $S_{\nu\nu}=0$. We prove this assertion now: Assume $T_{\alpha\beta}\neq 0$. We have to show $S_{\alpha\beta}=0$. The system (A.11) is invariant under equivalence transformations determined by orthogonal matrices performed simultaneously on S and T. Since T is real symmetric we may assume, without any loss of generality; that T is diagonal and $T_{11}\neq 0$. Now we put $\gamma=1$ in (A.11b) and obtain $S_{\alpha 1}=0$. Next we put $\beta=\gamma=1$ in (A.11a) and obtain $S_{\alpha\delta}=0$. This completes the proof of the assertion and, therefore, case (1) implies possibility (a) of the lemma.

In case (2) the following equations are satisfied:

$$\begin{aligned}&-S_{\mu\mu}\left(T_{\alpha\delta}\delta_{\beta\gamma}-T_{\alpha\gamma}\delta_{\beta\delta}-T_{\beta\delta}\delta_{\alpha\gamma}+T_{\beta\gamma}\delta_{\alpha\delta}\right)\\&+n\left(T_{\alpha\delta}S_{\beta\gamma}-T_{\alpha\gamma}S_{\beta\delta}-T_{\beta\delta}S_{\alpha\gamma}+T_{\beta\gamma}S_{\alpha\delta}\right)=0\end{aligned} \tag{A.12a}$$

$$-nS_{\alpha\mu}T_{\mu\gamma}+S_{\mu\mu}T_{\alpha\gamma}=0 \tag{A.12b}$$

$$S_{\mu\mu}\neq 0, \qquad T_{\nu\nu}=0 \tag{A.12c}$$

[Equation (A.12b) is, in fact, (A.9), and (A.12a) is a consequence of a substitution from (A.9) to (A.8).]

If $\{S_{\alpha\beta}, T_{\alpha\beta}\}$ form a solution of (A.12), it is easy to see that $\{S_{\alpha\beta}-(1/n)S_{\mu\mu}\delta_{\alpha\beta}, T_{\alpha\beta}\}$ form a solution of (A.11). It follows, therefore, as we know already, that $T_{\alpha\beta}=0$, which is possibility (a) of the lemma, or $T_{\alpha\beta}\neq 0$ and $S_{\alpha\beta}-(1/n)S_{\mu\mu}\delta_{\alpha\beta}=0$, which is possibility (b) of the lemma, [since (A.12c) is satisfied].

The treatment in case (3), ($T_{\nu\nu}\neq 0$, $S_{\mu\mu}=0$) is completely analogous to that of case (2): we have only to exchange the roles of S and T. This completes the proof of the lemma. ∎

The same type of proof implies a similar lemma in which S, T are Hermitean and R is unitary.

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